

where λ is a constant, our experimental result limits the total branching ratio of the K^+ into this channel to be less than 1.1×10^{-4} .

The vector-meson-dominant model^{8,9} and the η -pole model¹⁰ both predict branching ratios for $K^+ \rightarrow \pi^+ + \gamma + \gamma$ that are much lower than the upper limits which we have been able to set in this experiment.

We wish to thank E. Segrè for advice and encouragement, W. Hartsough and the Bevatron staff for an efficiently running machine, and E. McLeish for her effects at the scanning table.

†Work performed under the auspices of the U. S.

Atomic Energy Commission.

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POSITIVE DEFINITENESS OF GRAVITATIONAL FIELD ENERGY*

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(Received 20 September 1967)

The total gravitational field energy functional is shown to have only one extremum under variation of the metric field variables. At the extremum the energy vanishes and space is flat; second variation shows that the vacuum state is also a local minimum.

It has been increasingly recognized in recent years that the gravitational field, as described by general relativity, shares many of the basic physical properties of Lorentz-covariant field theories, particularly of massless gauge systems. Thus, asymptotically flat spaces, which describe isolated physical systems, can be assigned a well-defined total energy-momentum P^μ , satisfying the physical requirement that P^μ is covariant under Lorentz transformations and invariant under interior coordinate transformations.¹ The fact that the field is self-coupled, i.e., that gravitational field energy is itself a source of the gravitational field, is reflected in the nonlinear nature of the Einstein equations, particularly of the constraints that determine the energy. This is also the reason that the following fundamental problem had resisted solution until now²: Is the total energy of the gravitational field positive definite?³ The difficulty of this question is due to the implicit nature of the Hamiltonian of the theory—it can be given explicitly only as an

infinite series in the metric field variables. On the other hand, an affirmative answer is clearly essential for a satisfactory physical interpretation of the theory.

In this Letter we regard the energy as a functional of the gravitational field variables and consider its variational properties under change of geometry. We thereby establish that the functional has only one extremum, flat space. Further, the second variation about this “point”—the energy of a weak field—is shown to be positive. From these results, and to the extent that our functional behaves like a function of a finite number of variables,⁴ the positiveness problem is resolved in the affirmative: The vacuum (absence of any field excitations) is the lowest energy state, and, as a corollary, vanishing energy implies flatness. We restrict ourselves here to the source-free field, but the proof holds also in the presence of normal matter sources that have positive energy and are minimally coupled to gravitation. We consider here only “nonpathological” systems,

i.e., systems with Euclidean topology which can be reached from flat space by continuous deformation. Also, we assume that in such space at least one "minimal" hypersurface can be introduced.

We begin with the definition of the usual mass energy.⁵ For our purpose, it is adequate to take it as the flux integral,⁶

$$16\pi m = \oint dS_j (g_{ij,i} - g_{ii,j}) = -\oint dS_i (g g^{ij})_{,j}, \quad (1)$$

over a closed two-surface at spatial infinity.⁷ The value of m for any physical system is obtained from its definition (1) by solving the four Einstein constraint equations $G_\mu^0 = 0$ for the rel-

evant metric components. This is in complete analogy with electrodynamics, where the total charge Q is defined by $\oint \vec{E} \cdot d\vec{S}$, while its value for any system is obtained from the volume integral $\int d^3r \rho(\vec{r})$, because of the constraint $\nabla \cdot \vec{E} = \rho$. The Einstein constraints read⁸

$$R^0 \equiv \mathcal{R} - g^{-\frac{1}{2}} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) = 0, \quad R^i \equiv \nabla_j \pi^{ij} = 0, \quad (2)$$

where \mathcal{R} is the scalar curvature density of the hypersurface $t=0$, π^{ij} is a three-tensor density related to the extrinsic curvature of this hypersurface (it is essentially $\partial_0 g_{ij}$, the momentum conjugate to the field amplitude g_{ij}), and $\pi \equiv \pi_i^i$. By (1) and (2) we may thus write the total mass energy

$$\begin{aligned} 16\pi m &= -\int d^3r [\mathcal{R} - g^{-\frac{1}{2}} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) + (g_{ii,jj} - g_{ij,ij})] \\ &= -\int d^3r g^{\frac{1}{2}} [g^{ij} (\Gamma_{kl}^l \Gamma_{ij}^k - \Gamma_{ki}^l \Gamma_{lj}^k) - g^{-1} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2)]. \end{aligned} \quad (3)$$

Clearly we cannot simply vary $m[g_{ij}, \pi^{ij}]$ with respect to arbitrary changes $\delta_n g_{ij}$, $\delta_n \pi^{ij}$ of the variables, since the four nonlinear constraints of Eq. (2) must be respected under variation in order to compare only systems satisfying the Einstein equations. On the other hand, we cannot solve the R^0 constraint in closed form for a single "constraint" metric component. Instead, we add to $\delta_n g_{ij}$ a compensating $\delta_\epsilon g_{ij}$ to guarantee that $R^0 = 0$ remains satisfied. The key to our procedure is the fact that a $\delta_\epsilon g_{ij}$ can be found which actually leaves m unaffected. Consider the variations

$$\delta_T g_{ij} = \delta_n g_{ij} + 4\epsilon g_{ij}, \quad \delta_T \pi^{ij} = \delta \pi^{ij} \mathbf{T} + \delta \pi^{ij} \mathbf{L}, \quad (4)$$

where $\delta_n g_{ij}$ is a normal, arbitrary variation vanishing faster than $1/r$ at infinity, while ϵ may (and indeed must, as we shall see) vanish as slowly as $1/r$. The significance of the decomposition of $\delta_T g_{ij}$ is made apparent by writing

$$\delta_T g_{ij} \equiv \tilde{\delta} g_{ij} + \frac{1}{3} g_{ij} (\delta_T g_{lm}) g^{lm},$$

so that $\tilde{\delta} g_{ij}$ is clearly traceless and has five components, while ϵ is essentially the trace of $\delta_T g_{ij}$. The $\tilde{\delta} g_{ij}$ are then unconstrained, but yield five equations upon varying. For simplicity in text, we use six $\delta_n g_{ij}$'s instead, but the results are identical, with the redundant equation $R^0 = 0$ entering thereby. Similarly, we have divided $\delta_T \pi^{ij}$ into an arbitrary part $\delta \pi^{ij} \mathbf{T}$ which is divergenceless, and a longitudinal part $\delta \pi^{ij} \mathbf{L}$ which guarantees the maintenance of the constraints $R^i = 0$ under the full variation of Eq. (4). From the corresponding variations of the quantities occurring in (3),

$$\begin{aligned} -\delta_\epsilon \int \mathcal{R} d^3r &= -2 \int \epsilon \mathcal{R} d^3r + 8 \oint \nabla \epsilon \cdot d\vec{S}, \quad -\delta_\epsilon \int (g_{ii,jj} - g_{ij,ij}) d^3r = -8 \oint \nabla \epsilon \cdot d\vec{S}, \\ \delta_\epsilon \int (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) g^{-\frac{1}{2}} d^3r &= 2 \int \epsilon (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) g^{-\frac{1}{2}} d^3r, \end{aligned} \quad (5)$$

it is easy to see that $\delta_\epsilon m$ vanishes. Here the explicit metric dependence of the terms involving π^{ij} is fixed by the choice of the contravariant tensor density π^{ij} as the basic variable. The vanishing of $\delta_\epsilon m$ occurs only at the con-

straint $R^0 = 0$, and shows incidentally that the energy of a physical system in general relativity is a (three-space) conformal invariant.

To assure that the R^0 constraint remains sat-

ified, we demand that $\delta_T R^0 = 0$. Using the formulas

$$\delta_\epsilon R = 8\nabla_g^2 \epsilon - 4R\epsilon, \quad \delta\sqrt{g} = 6\epsilon\sqrt{g},$$

we find that ϵ must satisfy the Poisson equation⁹

$$8\nabla_g^2 \epsilon = -\delta_{(\delta_n g)} [R - g^{-1} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2)] + \delta_{(\delta_T \pi)} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2). \quad (6)$$

Here ∇_g^2 is the covariant Laplacian for the base metric g_{ij} . Note that the solution ϵ is indeed necessarily $O(1/r)$, and a linear functional of $\delta_n g_{ij}$, $\delta\pi^{ij}$. [An alternate derivation of $\delta_T m$ proceeds from its definition (1), which gives directly $\delta m = -8 \int \nabla^2 \epsilon d^3 r$, with $\nabla^2 \epsilon$ given by (6).] The variations $\delta\pi^{ij}$ must also be treated carefully as they must respect the R^i constraints (2). For this purpose we may invoke a recent explicit covariant solution¹⁰ of this transversality equation. One may write $\pi^{ij} = \pi^{ij} T + \pi^{ij} L$, where $\nabla_j \pi^{ij} T \equiv 0$. The variation of the transverse part $\delta\pi^{ij} T$ is thus free, while the three variations $\delta\pi^{ij} L$ are determined by the three conditions $\delta R^i = 0$ as solutions of Poisson-like equations in terms of δg_{ij} . As was the case for ϵ , these need not be solved explicitly, since

$$\delta_{(\delta\pi L)} m = (\frac{1}{8}\pi) \int d^3 r g^{-\frac{1}{2}} \pi^{ij} T \delta\pi_{ij} L = 0$$

(orthogonality of transverse and longitudinal tensors) on the "minimal" surface $\pi = 0$ whose existence we are assuming (though of course $\delta\pi \neq 0$).

The total variation of the mass energy is then effectively just

$$\delta_T m = (\delta_n g_{ij} \delta/\delta g_{ij} + \delta\pi^{ij} T \delta/\delta\pi^{ij}) m, \quad (7)$$

where the last set of variations can be treated as entirely unconstrained. From Eq. (3) the Euler-Lagrange equations for the extremum, $\delta_T m = 0$, are¹¹

$$R^{ij} - \frac{1}{2} g^{ij} R + O(\pi^2) = 0, \quad \pi^{ij} T = \pi^{ij} = 0. \quad (8)$$

Here $O(\pi^2)$ indicates terms quadratic in π^{ij} which will vanish anyhow because of the other Euler equations, $\pi^{ij} = 0$. Thus there is only one extremum which lies at the "point" $\pi^{ij} = 0 = R^i$ (taking into account the constraints $R^0 = 0 = R^i$). But this "point" is precisely flat space,¹² since the full curvature tensor $R_{\mu\nu\alpha\beta}$ vanishes when-

ever R_{ij} and π^{ij} vanish on a spacelike surface, and Einstein's equations hold.

Finally, we must show that the extremum is in fact an absolute minimum; this corresponds to evaluating second variations of the energy about flat space (weak-field excitations). Here one must take into account the first-order relations among the variations, particularly the value of $\delta\epsilon/\delta_n g_{ij}$ at the extremum. The term $\delta^2 m / \delta T g_{ij} \delta T g_{lm}$, which involves the curvature R , can be shown to be positive definite^{13,14} by explicit variation and use of Eq. (6). The mixed variations $\delta^2 m / \delta T g_{ij} \delta T \pi^{lm}$ clearly vanish at $\pi^{ij} = 0$, since m is quadratic in π^{ij} . This leaves $\delta^2 m / \delta T \pi^{ij} \delta T \pi^{lm}$, where m must include the variations $-\frac{1}{2} \int d^3 r g^{-1/2} \delta\pi \delta\pi$ even though $\pi = 0$. However, since $\nabla_j \delta\pi^{ij} = 0$ at $\pi^{ij} = 0$, we may fall back on the fact that at flat three-space where rectangular coordinates ($g_{ij} = \delta_{ij}$) are allowed, the form

$$\int d^3 r g^{-\frac{1}{2}} [\delta_T \pi^{ij} \delta_T \pi_{ij} - \frac{1}{2} (\delta\pi)^2]$$

is positive if $\partial_j \delta_T \pi^{ij} = 0$. (This integral is simply the kinetic energy in the linearized approximation.) Thus the energy of weak systems is strictly positive,¹⁵ and in fact has the transparent form

$$\frac{1}{2} \delta_T^2 m_{\text{flat}} = \int d^3 r [\frac{1}{4} (\nabla_n \delta_T g^{\text{TT}})^2 + (\delta\pi^{\text{TT}})^2] \geq 0 \quad (9)$$

appropriate to a massless spin-2 particle, whose variables are transverse traceless (TT) in the flat-space sense.

Our considerations hold for isolated systems, i.e., for asymptotically flat spaces [$g_{\mu\nu} \sim \eta_{\mu\nu} + O(1/r)$, $\partial_\alpha g_{\mu\nu} \sim O(r^{-2})$ at $t=0$] with Euclidean topology, and we have assumed that the Poisson-type equation (6) (and a similar one for $\delta\pi^L$) have global solutions. Also, it is implicit in our variational process that we compare only spaces which lie on the same functional "branch" as flat space, and omit from consideration any possible "pathological" spaces which do not reduce smoothly to flat space in some suitable limit, such as a limit of decreasing field intensity. (It is probably only for such systems that the notion of energy is at all applicable or relevant.)

If, further, our variational results on m can be taken to imply that this functional is positive,⁴ they have the following implications: The "free-field" energy of the full classical Einstein theory has zero as its lowest value.

This value is reached only by the vacuum state, where there are no physical excitations (flat space), and the single condition $m=0$ is equivalent to the vanishing of the full curvature tensor $R_{\mu\nu\alpha\beta}$. These properties coincide with those of all Lorentz-covariant systems considered in the rest of physics, and have obvious relevance to quantization.

*Work supported in part by the National Aeronautics and Space Administration under Grant No. NSG 163-61 and by the U. S. Air Force, Office of Aerospace Research, under Grant No. AF 368-67.

¹R. Arnowitt, S. Deser, and C. W. Misner, *Phys. Rev.* **122**, 997 (1961); R. Arnowitt and S. Deser, to be published.

²Positiveness of the energy has been established for various special classes of solutions: H. Araki, *Ann. Phys. (N.Y.)* **7**, 456 (1959); D. R. Brill, *ibid.* **7**, 466 (1959); R. Arnowitt, S. Deser, and C. W. Misner, *ibid.* **11**, 116 (1960). The last paper also mentions some physical arguments for positiveness in general.

³Unlike the situation in Lorentz-covariant theories, it is well known that the zero point of energy is not arbitrary in general relativity; m must be zero for flat space.

⁴For functionals, however, positiveness does not always follow from knowledge that there is only one extremum which is a local minimum. Existing mathematical criteria do not yet seem directly applicable to our problem.

⁵We use the historical notation m for the energy (from the Schwarzschild solution mass). Alternatively, m is also the invariant mass in c.m. frames, where $\vec{p}=0$ (which is probably always allowed for bounded systems). The units are $16\pi\gamma=1=c$; Latin indices range 1, 2, 3; the signature is (+++).

⁶All the usual definitions of gravitational energy agree with Eq. (1) when $g_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu}$ approach their flat space values as $O(r^{-1})$ and $O(r^{-2})$, respectively, at infinity. For a more precise discussion of the definition and invariance of gravitational field energy, see Ref. 1.

⁷Of course, $dm/dt=0$ by the Einstein equations.

⁸We follow the notation of the last paper of Ref. 2. All quantities in this Letter (except $R_{\mu\nu\alpha\beta}$) refer to

the three-space $t=0$ with metric g_{ij} . Thus, ∇ denotes covariant differentiation in this metric, and $g \equiv \det g_{ij}$. The sign convention on R_{ij} is $R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j}$ at $\Gamma=0$.

⁹We assume that the various Poisson equations we need have global solutions; alternatively, we are restricting attention to spaces for which this is the case. No attempt is being made here to solve the unsolved mathematical problems of rigor connected with such equations in curved space.

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¹¹The first part of Eq. (8) is obvious since it is the three-dimensional analog of the standard Einstein variational principle. The second part, $\pi^{ij}=0$, results because the kinetic energy variation has only one term, $\int d^3r g^{-\frac{1}{2}} \pi^{ij} T_{\delta\pi ij}$, when $\pi=0$ and $\pi^{ij}L=0$ (due to $\nabla_j \pi^{ij}=0$). The "cross terms" $\int d^3r g^{-\frac{1}{2}} \pi^{ij} T_{\delta\pi ij} L$ vanish since T and L tensors are orthogonal.

¹²For a detailed proof, see, for example, R. Arnowitt and S. Deser, *Ann. Phys. (N.Y.)* **23**, 319 (1963). The existence of a flat surface of vanishing extrinsic curvature is a sufficient condition for flatness, but it is not necessary, since curved surfaces can also be imbedded in flat space.

¹³Positiveness of weak-field energy when $\pi^{ij}=0$ was first demonstrated by Araki (Ref. 2).

¹⁴In studying second variations of m , one must check that second variations of the constraints can also be made to vanish. This is indeed the case, but details of this are unnecessary in evaluating $\delta^2 m$, just as details of how δR^0 vanishes were unnecessary in evaluating $\delta T m$.

¹⁵If the "minimal" surface condition is lifted, it is known (Ref. 10) that the kinetic energy is not a positive form (unless $R_{ij}=0$). Physically, however, the theorem still holds, for the value of π just fixes the time coordinate, and transformations leading from arbitrary π to $\pi=0$ exist (at least formally), leaving the value of m invariant. When $\pi \neq 0$, the method in text no longer leads to Eq. (8); one obtains an expression for $\delta T m$ depending explicitly on the constraint variations. We have not been able to determine the corresponding Euler-Lagrange equations, but they are presumably that generalization of (8) which expresses the vanishing of $R_{\mu\nu\alpha\beta}(g, \pi)$ on an arbitrary spacelike surface (the Gauss-Codazzi equations). We are indebted to C. W. Misner for enlightening discussion of this question.